Electromagnetic Fields

Part 2

Lec 02

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2.8 Capacitors

2.8.1 Theory of Capacitors

- A capacitor is an electrical device composed of two conductors which are separated through a dielectric medium and which can store equal and opposite charges. This states that all flux lines leaving one conductor must necessarily terminate at the surface of the other conductor. The conductors are referred to as the plates of the capacitor. The plates may be separated by free space or a dielectric as shown in Fig. 2.22.
- Consider the two-conductor capacitor of Fig. 2.22. The conductors are maintained at a potential difference V given by:

$$V = V_1 - V_2 = -\int_2^1 \vec{E}.d\vec{l}$$

- Where $\vec{E}$ is electric field existing between the conductors and conductor 1 is assumed to carry positive charge.

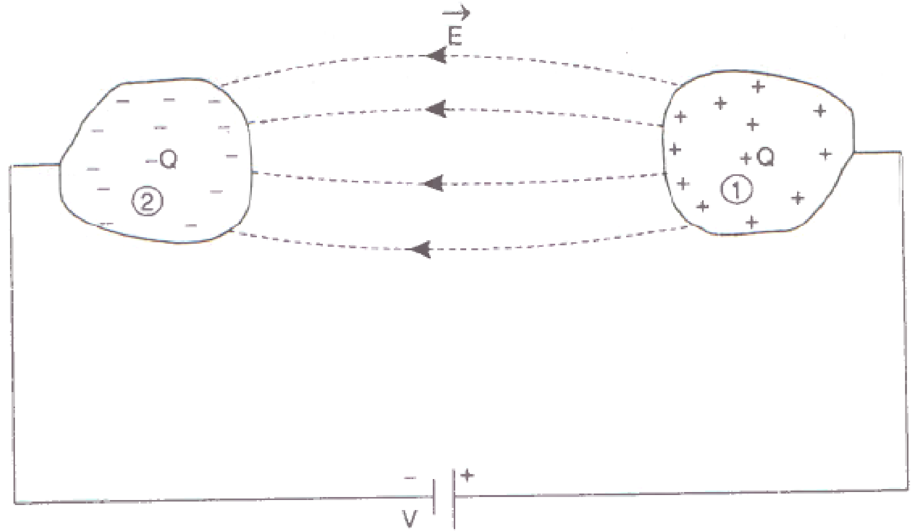


Fig. 2.22 Two-conductor capacitor

2.8 Capacitors (Continued)

2.8.1 Theory of Capacitors

- The capacitance, C, of the capacitor is the ratio of the magnitude of the charge on one of the plates to the potential difference between them. That is:

$$C = \frac{Q}{V} = \frac{\oint_S \vec{E}.d\vec{S}}{\int_L \vec{E}.d\vec{l}} \quad Farad\ [F] \tag{2.46}$$

- The negative sign before $V = -\int \vec{E}.d\vec{l}$ has been dropped because only absolute value of V is to be considered. The capacitance of the capacitor is measured in Farad.
- The capacitance, C can be obtained for any given two-conductor capacitance by assuming Q and determining V in terms of Q. This method involves Gauss's law.
- The following steps are used to find the capacitance of any two-conductor capacitance:

I. Choose a suitable coordinate system.
2. Let the conducting plates carry charges +Q and -Q.
3. Determine $\vec{E}$ using Coulomb's law or Gauss's law.
4. Finally obtain C from C = Q / V .

2.8 Capacitors (Continued)

2.8.2 Capacitance of Parallel Plate Capacitor

- Consider the parallel plate capacitor in Fig. 2.23. Each of the plates has an area A (m^2) and the; separated by a distance d (m). Assume the plates 1 and 2 carry charges +Q and -Q uniformly distributed on them, so that:

$$\rho_s = \frac{Q}{A} \ C/m^2 \tag{2.47}$$

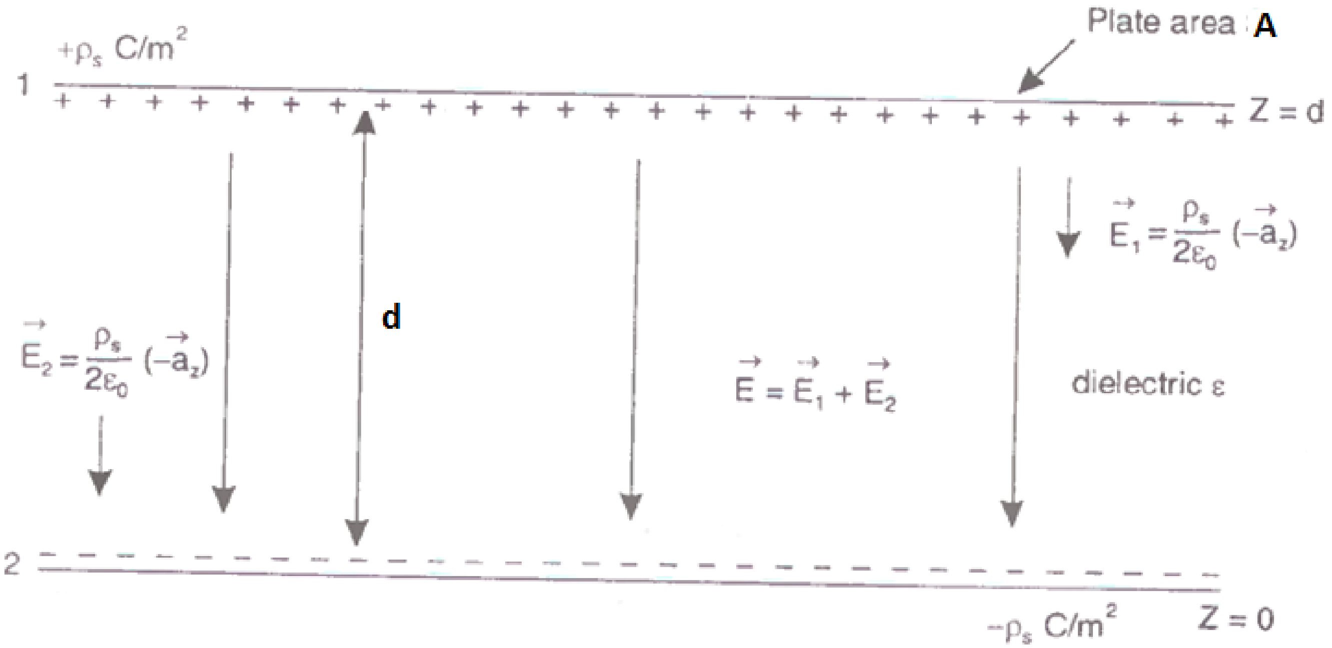


Fig. 2.23 Parallel plate capacitor

2.8 Capacitors (Continued)

2.8.2 Capacitance of Parallel Plate Capacitor

- An ideal parallel plate capacitor is one in which the plate separation d is very small compared with the dimensions of the plate. Assuming such an ideal case, the fringing field at the edge of the plate be ignored so that the field between them is considered uniform. If the space between the plate filled with homogeneous dielectric with permittivity ε.
- Electric field intensity $\vec{E}$ at any point in between two plates is found by Gauss's law. Its value at a point below the surface charge sheet of $+\rho_s$ is given by (2.25) as:

$$\vec{E}_1 = \frac{\rho_s}{2\varepsilon}\vec{a}_n = \frac{\rho_s}{2\varepsilon}(-\hat{a}_z) \qquad (2.48a)$$

Here, the medium is not free space so, $\varepsilon = \varepsilon_r \varepsilon_o$. Similarly, electric field intensity at a point above the surface charge sheet of $-\rho_s$ is given by:

$$\vec{E}_2 = \frac{\rho_s}{2\varepsilon}\vec{a}_n = \frac{-\rho_s}{2\varepsilon}\hat{a}_z \qquad (2.48b)$$

- Total electric field intensity between the two plates is given by:

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = \frac{\rho_s}{\varepsilon}(-\hat{a}_z) \qquad (2.49a)$$

- From (2.47), $\rho_s = Q / A$, substitute in (2.49) we get:

$$\vec{E} = \frac{Q}{\varepsilon A}(-\hat{a}_z) \qquad (2.49a)$$

2.8 Capacitors (Continued)

2.8.2 Capacitance of Parallel Plate Capacitor

- The potential difference between the two plates, V is given by:

$$V = -\int_{2}^{1} \vec{E}.d\vec{l} = -\int_{z=0}^{z=d} \frac{Q}{\varepsilon A}(-\hat{a}_z).dz\hat{a}_z = \frac{Qd}{\varepsilon A} \qquad (2.50)$$

- Therefore, the value of the capacitance is given by:

$$C = \frac{Q}{V} = \frac{\varepsilon A}{d} \;\; [F] \qquad (2.51)$$

- From eqn. (2.45) the energy stored in the parallel plate capacitor is:

$$W_E = \frac{1}{2}\int_v \varepsilon E^2 dv = \frac{1}{2}\int_v \varepsilon\left(\frac{Q}{\varepsilon A}\right)^2 dv = \frac{1}{2}\frac{Q^2}{\varepsilon A^2}\int_v dv = \frac{1}{2}\frac{Q^2}{\varepsilon A^2}(Ad) \qquad (2.52)$$

$$= \frac{1}{2}\frac{Q^2 d}{\varepsilon A} = \frac{1}{2}\frac{Q^2}{(\varepsilon A/d)} = \frac{1}{2}\frac{Q^2}{C} \;\; [J]$$

- Different forms of the energy stored in the parallel plate capacitor are:

$$W_E = \begin{cases} \frac{1}{2}\frac{Q^2}{C} \\ \frac{1}{2}QV \\ \frac{1}{2}CV^2 \end{cases} \;\; [J] \qquad (2.53)$$

2.8 Capacitors (Continued)

2.8.2 Capacitance of Parallel Plate Capacitor

Example 2.13:

Calculate the capacitance of a parallel plate capacitor having mica dielectric with $\varepsilon_r = 6$, a plate area A = 10 in^2 and a separation d= 0.01 in. What is the energy stored in this capacitor if it is charged to a voltage, V = 5 V.

Solution:

The value of the capacitance, C is given by (2.51):

$$C = \frac{\varepsilon A}{d} = \frac{6x8.85x10^{-12}\,x(10x2.54x10^{-2})^2}{0.01x2.54x10^{-2}} = 1.349\ nF$$

The energy stored in the parallel plate capacitor is given by (2.53):

$$W_E = \frac{1}{2}CV^2 = \frac{1}{2}x(1.349x10^{-9})x25 = 16.8625\ nJ$$

2.8 Capacitors (Continued)

2.8.3 Capacitance of Coaxial Capacitor

- Consider two coaxial conductors of length, L [m] having inner radius, a and outer radius, b shown in Fig. 2.24.
- Let the space between the conductors be filled with a homogeneous dielectric with permittivity, ε. Assume the conductors I and 2 carry +Q and -Q coulomb of charges uniformly distributed on them.

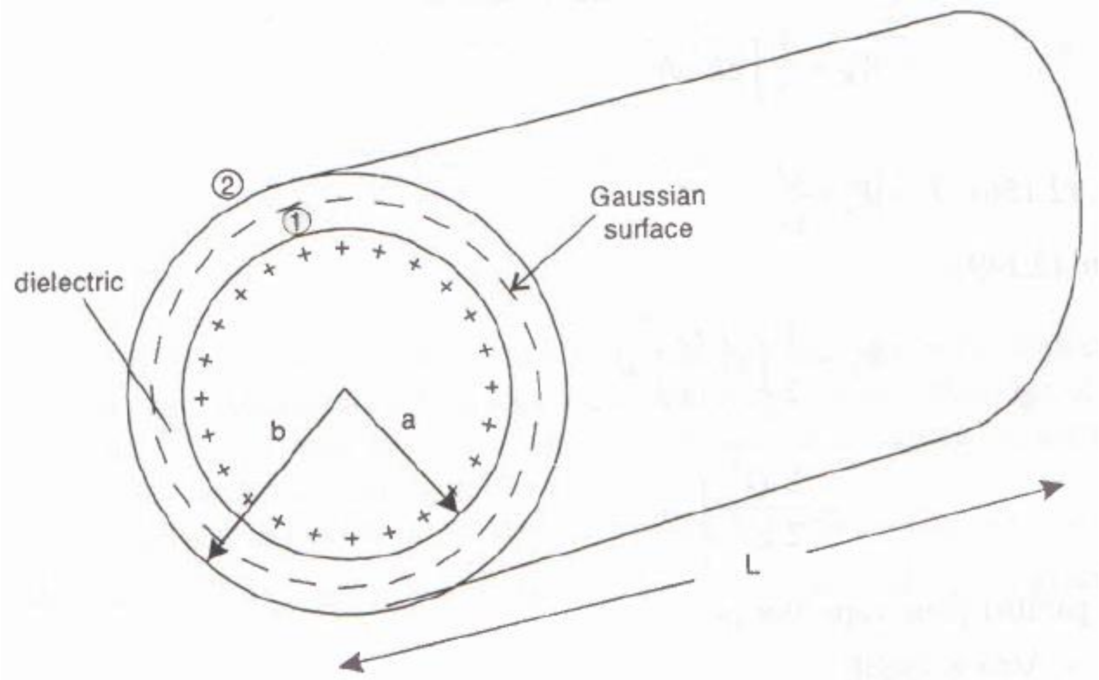


Fig. 2.24 Coaxial capacitor

2.8 Capacitors (Continued)

2.8.3 Capacitance of Coaxial Capacitor

- For convenience, assume cylindrical coordinates system. The linear charge density ρ_L is given as:

$$\rho_L = \frac{Q}{L} \ C/m \qquad (2.54)$$

- Electric field intensity, $\vec{E}$ between two coaxial conductors is obtained by applying Gauss's law to an arbitrary Gaussian cylindrical surface of radius p. From eqn. (2.23), we have:

$$\vec{E} = \frac{\rho_L}{2\pi\varepsilon\rho}\hat{a}_\rho = \frac{Q}{2\pi\varepsilon\rho L}\hat{a}_\rho \ V/m \quad (a < \rho < b) \qquad (2.55)$$

- From (2.34) and neglecting flux fringing at the cylindrical ends, we get:

$$V = -\int_2^1 \vec{E}.d\vec{l} = -\int_{\rho=b}^{\rho=a} E\hat{a}_\rho.d\rho\hat{a}_\rho = -\int_{\rho=b}^{\rho=a} \frac{Q}{2\pi\varepsilon\rho L}.d\rho = \frac{Q}{2\pi\varepsilon L}\ln(b/a) \qquad (2.56)$$

- Thus the capacitance of coaxial cylinder is

$$C = \frac{Q}{V} = \frac{2\pi\varepsilon L}{\ln(b/a)} \ [F] \qquad (2.57)$$

2.8 Capacitors (Continued)

2.8.3 Capacitance of Coaxial Capacitor

Example 2.14:

What is the capacitance / kilometer of length of air-filled coaxial cable with an inner conductor of 3 mm diameter and an outer conductor of inside diameter of 1 cm.

Solution:

The capacitance / meter of length of the line, C is given by (2.57):

$$C = \frac{Q}{V} = \frac{2\pi\varepsilon L}{\ln(b/a)} \;[F] \;\Rightarrow\; = \frac{C}{L} = \frac{2\pi\varepsilon}{\ln(b/a)} \;[F/m]$$

$$\frac{C}{L} = \frac{2\pi\varepsilon_o}{\ln(b/a)} = \frac{2\pi x 8.85 x 10^{-12}}{\ln(0.5x10^{-2}/1.5x10^{-3})} x1000 = 0.4 \;[\mu F/km]$$

2.8 Capacitors (Continued)

2.8.4 Capacitance of a Spherical Capacitor

- Consider two concentric spherical conductors of inner radius, a and outer radius, b separated by a dielectric medium with permittivity, ε as shown in Fig. 2.25. Assume charges +Q and -Q are on the inner and outer spheres respectively.
- Applying Gauss's law to an arbitrary Gaussian spherical surface of radius r (a < r < b), we get:

$$Q = \varepsilon \oint_S \vec{E}.d\vec{S} = \varepsilon \oint_S E\hat{a}_r.dS\hat{a}_r = \varepsilon E \oint_S dS$$

$$\vec{E} = \frac{Q}{4\pi\varepsilon r^2}\hat{a}_r \; V/m \qquad (a < r < b)$$

$$V = -\int_2^1 \vec{E}.d\vec{l} = -\int_{r=b}^{r=a} \frac{Q}{4\pi\varepsilon r^2}\hat{a}_r.dr\hat{a}_r$$

$$V = \frac{Q}{4\pi\varepsilon}\left(\frac{1}{a} - \frac{1}{b}\right)$$

Thus the capacitance of spherical capacitor is:

$$C = \frac{Q}{V} = \frac{4\pi\varepsilon}{\left(\frac{1}{a} - \frac{1}{b}\right)} \; [F] \qquad (2.58)$$

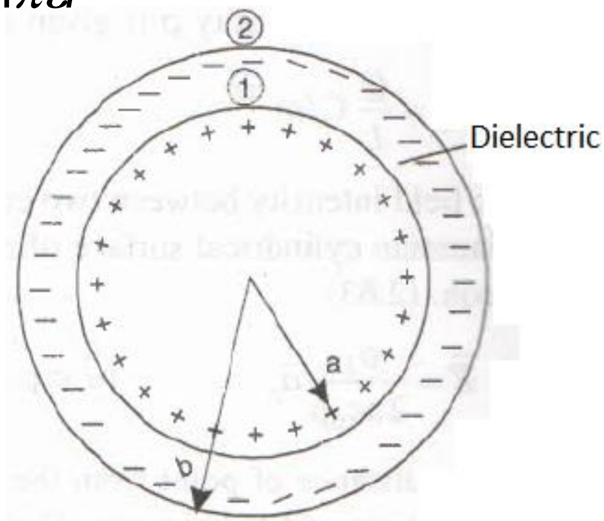


Fig. 2.25 Spherical capacitor

2.8 Capacitors (Continued)

2.8.4 Capacitance of a Spherical Capacitor

Example 2.15:

The radii of spherical capacitor are 20 cm and 5 cm and the space between two spheres is filled with an insulator of $\varepsilon_r = 2$. Find:

(a) The sphere capacitance.

(b) The greatest electric intensity (E_{max}) in the dielectric when the potential difference between the sphere is 20 kV.

Solution:

(a) The value of the capacitance, C is given by (2.58):

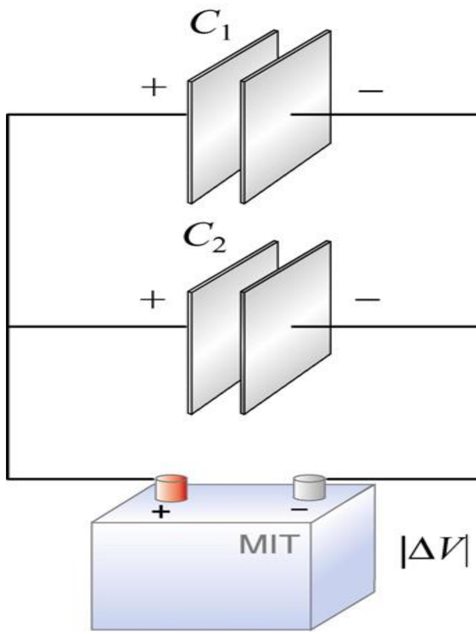
$$C = \frac{4\pi\varepsilon}{\left(\frac{1}{a} - \frac{1}{b}\right)} = \frac{4\pi x 2 x 8.85 x 10^{-12}}{\left(\frac{1}{5x10^{-2}} - \frac{1}{20x10^{-2}}\right)} = 7.41\ [pF]$$

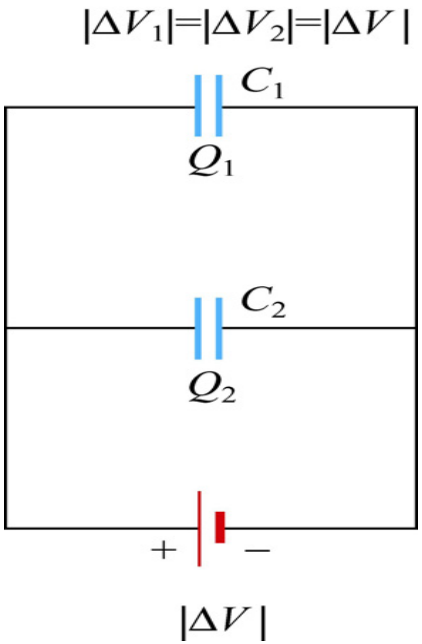
(b) From the equation $\vec{E} = \frac{Q}{4\pi\varepsilon r^2}\hat{a}_r\ V/m$ and $V = \frac{Q}{4\pi\varepsilon}\left(\frac{1}{a} - \frac{1}{b}\right)$ we get:

$$E = \frac{V}{r^2\left(\frac{1}{a} - \frac{1}{b}\right)_r} \quad \Rightarrow \quad E_{\max}\big|_{r=a} = \frac{20\ kV}{5^2\left(\frac{1}{5} - \frac{1}{20}\right)} = 5.33\ kV/cm$$

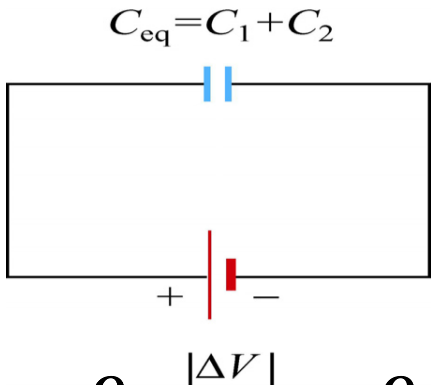
CAPACITORS IN ELECTRIC CIRCUITS

(1) Parallel Connection:





Same Voltage on all the capacitors



$$C_1 = \frac{Q_1}{|\Delta V|} \quad , \quad C_2 = \frac{Q_2}{|\Delta V|}$$

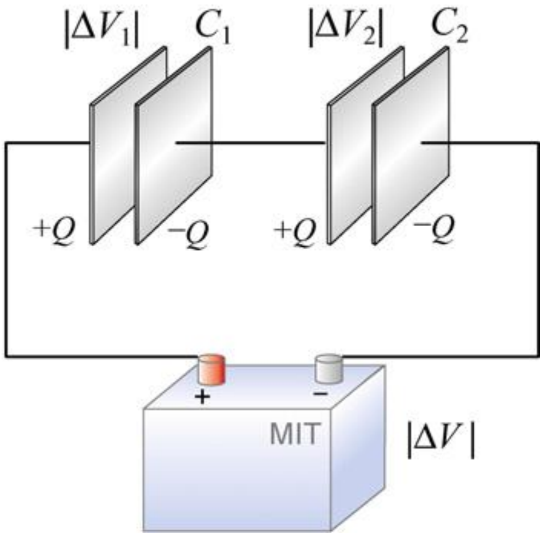
$$Q = Q_1 + Q_2 = C_1|\Delta V| + C_2|\Delta V| = (C_1 + C_2)\,|\Delta V|$$

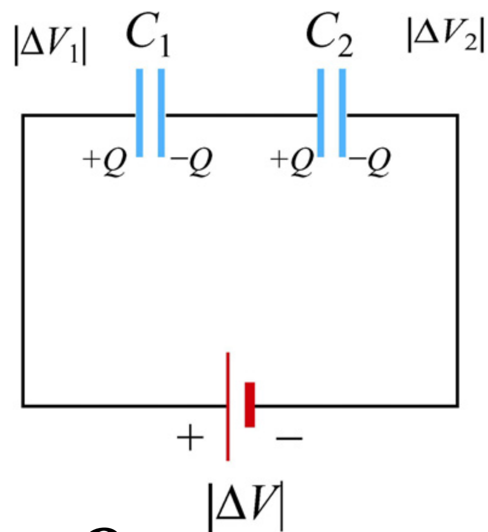
$$\therefore C_{eq} = C_1 + C_2 = \frac{Q}{|\Delta V|}$$

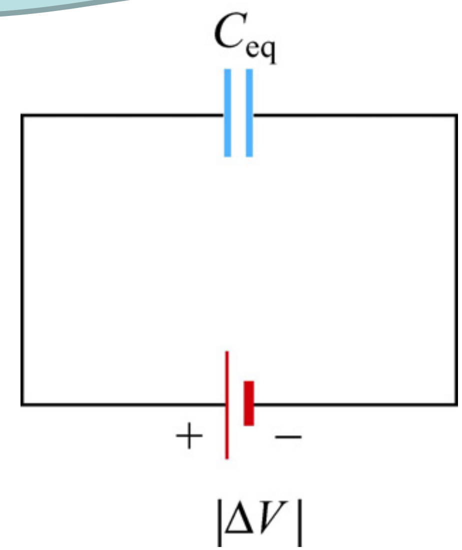
$$\therefore C_{eq} = C_1 + C_2 + \dots + C_N = \sum_{i=1}^{N} C_i$$

(2) Series Connection:

Same Q passes in all the capacitors







$$|\Delta V_1| = \frac{Q}{C_1} \quad , \quad |\Delta V_2| = \frac{Q}{C_2} \quad \Rightarrow \quad \therefore |\Delta V| = |\Delta V_1| + |\Delta V_2|$$

$$\frac{Q}{C_{eq}} = \frac{Q}{C_1} + \frac{Q}{C_2} \quad \Rightarrow \quad \therefore \frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$$

$$\therefore \frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \dots + \frac{1}{C_N} = \sum_{i=1}^{N} \frac{1}{C_i}$$

STACKED DIELECTRIC CAPACITORS

(1) Plate Capacitor:

$$V = V_1 + V_2$$

$$Q_1 = Q_2 = Q$$

$$C_1 = \frac{\varepsilon_o \varepsilon_{r1} A}{d/2}$$

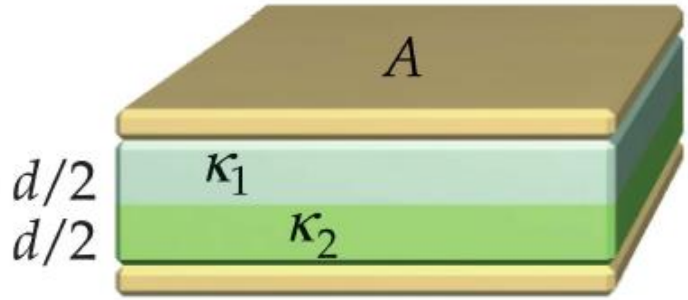
$$C_2 = \frac{\varepsilon_o \varepsilon_{r2} A}{d/2}$$

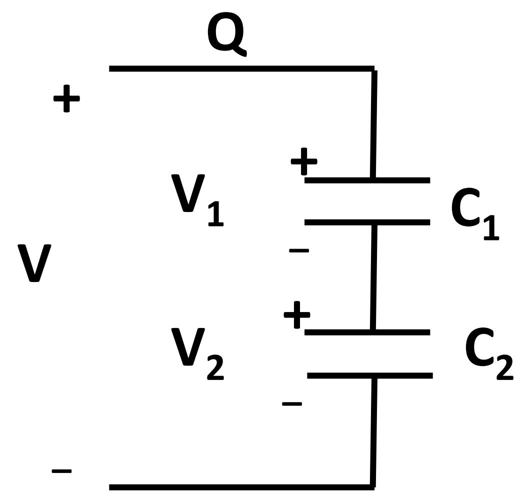
$$C_{eq} = \frac{C_1 C_2}{C_1 + C_2} = \frac{Q}{V}$$

$$\therefore V_1 = \frac{Q}{C_1} \quad \& \quad V_2 = \frac{Q}{C_2}$$

$$E_1 = \frac{V_1}{d_1} \quad \& \quad E_2 = \frac{V_2}{d_2}$$

$$W_E = \frac{1}{2} C V^2$$





$$V = V_1 = V_2$$

$$Q = Q_1 + Q_2$$

$$C_1 = \frac{\varepsilon_o \varepsilon_{r1} A/2}{d}$$

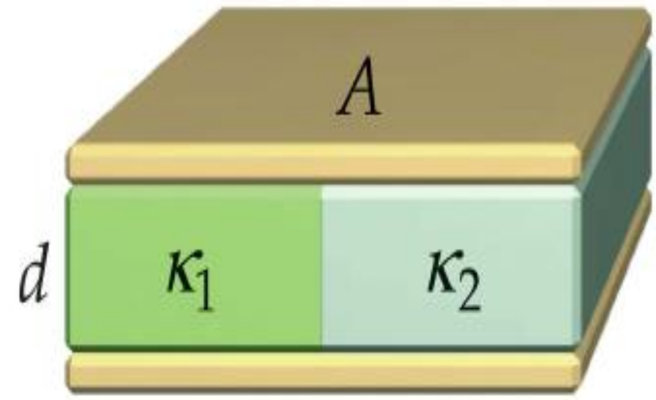
$$C_2 = \frac{\varepsilon_o \varepsilon_{r2} A/2}{d}$$

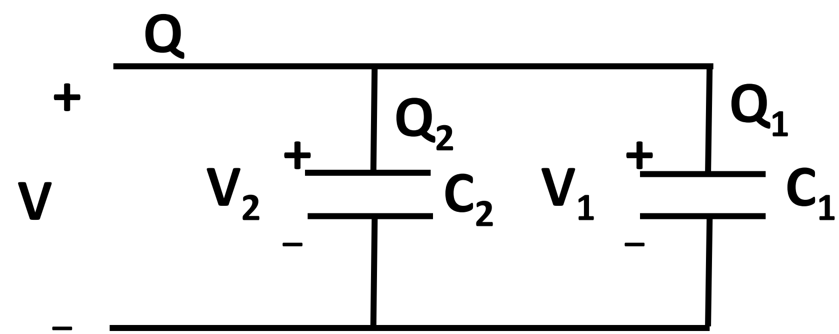
$$C_{eq} = C_1 + C_2 = \frac{Q}{V}$$

$$\therefore Q_1 = C_1 V \quad \& \quad Q_2 = C_2 V$$

$$E_1 = E_2 = \frac{V}{d}$$

$$W_E = \frac{1}{2} C V^2$$





Thank you for your attention

Assoc. Prof. Dr. Moataz Elsherbini